FINITISTIC DIMENSION CONJECTURES

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INTRODUCTION

The left little finitistic dimension of an algebra Λ is the supremum of the projective dimensions of the finitely generated left Λ-modules having finite projective dimension, and the left big finitistic dimension of an algebra Λ is the supremum of the projective dimensions of all left Λ-modules having finite projective dimension. For a finite dimensional algebra Λ these dimensions will be denoted by $\ell\text{fin.dim}\,\Lambda$ and $\ell\text{Fin.dim}\,\Lambda$, respectively.

The notion of the (little) finitistic dimension of an algebra was introduced by Maurice Auslander and David Buchsbaum in [AB] for commutative local noetherian rings where they proved that for a commutative local noetherian ring the little finitistic dimension is equal to maximal length of a regular sequence in the maximal ideal. Then Irving Kaplansky considered the supremum of the projective dimensions over all modules of finite projective dimension, arriving at the notion of the big finitistic dimension. For a commutative noetherian ring this dimension coincides with the Krull dimension of the ring. That the big finitistic dimension is at least the Krull dimension for this class of rings was proved by Bass in 1962 [B2] and the proof of the equality was completed by Gruson and Raynaud in 1971 [GR]. Kaplansky also characterized those commutative rings where the big finitistic dimension vanishes at a seminar at the university of Chicago in 1958. This was the background for the paper [B1] by Bass where the finitistic dimension conjectures were first formulated as questions raised by Rosenberg and Zelinsky. Question one: Is it true that the little and big finitistic dimension coincide? Question two: Is the little finitistic dimension finite?

For a commutative noetherian local ring one has that the two finitistic dimensions coincide if and only if the depth and the Krull dimension coincide, which is usually expressed by saying that the ring is Cohen-Macauley. However, not all commutative noetherian local rings are Cohen-Macauley and therefore the two finitistic dimensions do not coincide for commutative noetherian local rings in general.

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Also the finitistic dimension of a commutative noetherian ring is not necessarily finite. Her one can use Nagata's first example of a commutative noetherian ring with infinite Krull dimension. Let k be a field, $k[X_1, X_2, \cdots, X_n, \cdots]$ be the polynomial ring in infinitely many indeterminates, S the intersection of the complements of the prime ideals I_i, generated by the indeterminates $X_{2^i}, X_{2^i+1}, \cdots X_{2^{i+1}-1}$, $i = 1, 2, \cdots$. Then the ring $R = k[X_1, X_2 \cdots]_S$ obtained by localizing the polynomial ring in S is noetherian and has infinite finitistic dimension.

The first example of a finite dimensional algebra Λ where the two dimensions do not coincide was given in 1992 by Birge Huisgen-Zimmermann [HZ1]. The examples given in [HZ1] consist of a family of monomial relation algebras. A more accessible example of such algebras was given by me in 1996. This example may be interpreted as a repeated one point extension of basically injective modules starting with the truncated polynomial algebra in two indeterminates.

However, question two is still unanswered for finite dimensional algebras.

Main Topic

In 1991, Auslander and Reiten proved that the finitistic dimension was finite for an algebra if the embedding of the category of finitely generated modules of finite projective dimension into the category of all finitely generated modules is "nice". Her "nice" means that the subcategory of modules of finite projective dimension is contravariantly finite in the category of finitely generated modules. Birge Huisgen-Zimmermann and I proved later that this "nice" embedding property would also imply that the little and big finitistic dimension coincide. Therefore, examples where the embedding does not satisfy this "nice" property appear frequently.

Also in 1991 E. Green and B. Huisgen-Zimmermann proved that each algebra with vanishing radical cube, has finite finitistic dimension, and P. Dräxler and D. Happel proved the generalized Nakayama conjecture for algebras Λ of Loewy length $2n+1$ where Λ modulo the n-th power of the radical is of finite representation type. i.e. for each simple Λ-module S, there is an $i \geq 0$ such that $\operatorname{Ext}^i_\Lambda(S, \Lambda) \neq 0$.

Using some general principles, K. Igusa and G. Todorov provided in a couple of pages the necessary tools Y. Wang needed to generalize both these results to the following.

Theorem 1. (Wang) *If Λ is an artin algebra and I is an ideal such that Λ/I is of finite representation type, then* $\sup\{\operatorname{pdim} M \mid M \textit{ is finitely generated}, \operatorname{pdim} M < \infty,\ I^2M = 0\} < \infty$.

The idea behind this is a clever use of Fitting's lemma and of the fact that the ring of integers is noetherian.

Lemma 2. (i) *Let W be any module over a noetherian ring R and let $f : W \to W$ be an endomorphism of W.*

(a) *For any finitely generated submodule V of W there is a (smallest) integer $n_f(V)$ such that $f^i : f^{n_f(V)}(V) \to f^{n_f(V)+i}(V)$ is an isomorphism for all $i \geq 0$.*

(b) *If $U \subseteq V$ are finitely generated submodules of W, then $n_f(U) \leq n_f(V)$.*

(c) (Fitting's lemma) *If W is of finite length, then $W = \operatorname{Im} f^{n_f(W)} \amalg \operatorname{Ker} f^{n_f(W)}$.*

Next, for an artin algebra Λ, consider the abelian group K generated by the symbols $[C]$ where C ranges over the f.g. Λ-modules, modulo the relations

(i) $[C] = [A] + [B]$ if $C \simeq A \amalg B$,

(ii) $[C] = 0$ if $\operatorname{pdim} C < \infty$.

Then K is the free abelian group ((0) if $\operatorname{gl.dim} \Lambda < \infty$) on the isomorphism classes of f.g. indecomposable Λ-modules of infinite projective dimension. For each f.g. module M let $L[M] = [\Omega(M)]$ where $\Omega(M)$ is the first syzygy of M. Since Ω commutes with direct sums and takes modules of finite projective dimension to modules of finite projective dimension, L induces a group homomorphism $L : K \to K$. For a finitely generated module M, let $\langle M \rangle$ denote the subgroup of K generated by the indecomposable direct summands of M. Then $\langle M \rangle$ is a finitely generated subgroup of K and the number $n_L(\langle M \rangle)$ from the previous Lemma with $R = \mathbb{Z}$, $W = K$ and $f = L$, is written $n_L(M)$. It satisfies the following properties:

(i) $n_L(M \amalg M) = n_L(M)$,

(ii) $n_L(M) \leq n_L(M \amalg N)$.

For a finitely generated module M, denote by $\operatorname{fin.dim} M$ the number
$\sup\{\operatorname{pdim} N | N \text{ a direct summand of } M \text{ and } \operatorname{pdim} N < \infty\}$.

The key result now states:

Proposition 3. (Igusa, Todorov) *Suppose that $0 \to A \to B \to C \to 0$ is a short exact sequence of finitely generated modules over an artin algebra Λ with* $\operatorname{pdim} C < \infty$. *Then* $\operatorname{pdim} C \leq n_L(A \amalg B) + \operatorname{fin.dim}(\Omega^{n_L(A\amalg B)}(A \amalg B)) + 1$.

Proof: Observe that if $\operatorname{pdim}(A \amalg B) < \infty$ then $n_L(A \amalg B) = 0$, and $\operatorname{pdim} C \leq 0 + \operatorname{fin.dim} \Omega^0(A \amalg B) + 1 = \operatorname{pdim}(A \amalg B) + 1$ which is the usual formula for the relations between the projective dimensions of the modules in an exact sequence. Therefore I may assume that $\operatorname{pdim}(A \amalg B) = \infty$.

Since C has finite projective dimension, $\Omega^s(A) \simeq \Omega^s(B)$ for some integer s. Therefore $[\Omega^s(A)] = [\Omega^s(B)]$ in K, and hence $L^s([A]-[B]) = 0$. But $[A]-[B]$ is an element of $\langle A \amalg B\rangle$, and therefore if some power of L annihilates this element, it is already annihilated by L^{n_L} where I for short write, from now on, n_L for $n_L(A \amalg B)$. Applying Ω^{n_L} to the exact sequence, one therefore obtains an exact sequence

$$0 \longrightarrow X \amalg P \xrightarrow{\left(\begin{smallmatrix} f & g \\ h & j \end{smallmatrix}\right)} X \amalg Q \longrightarrow \Omega^{n_L} C \longrightarrow 0$$

for some module X without indecomposable direct summands of finite projective dimension, and modules P and Q of finite projective dimension. Here $f : X \to X$ is an endomorphism, which by Fitting's lemma decomposes X as $\operatorname{Ker} f^t \amalg \operatorname{Im} f^t$ for some t. Let s be the projective dimension of $P \amalg Q$. Since $\operatorname{Ext}^{s+i}(P,\ ) = 0 = \operatorname{Ext}^{s+i}(Q,\ )$ for all $i \geq 1$, I have that $\operatorname{Ext}^{s+i}(\left(\begin{smallmatrix} f & g \\ h & j \end{smallmatrix}\right),\ ) = \operatorname{Ext}^{s+i}(f,\ )$ and therefore also $(\operatorname{Ext}^{s+i}(\left(\begin{smallmatrix} f & g \\ h & j \end{smallmatrix}\right),\ ))^t = (\operatorname{Ext}^{s+i}(f,\ )^t$ which again is equal to $\operatorname{Ext}^{s+i}(f^t,\ )$ for all $i \geq 1$. Observe also that $\operatorname{Ker} f^t \simeq \operatorname{Coker} f^t$ and that this module is of infinite projective dimension if it is nonzero. Now, if f is not an isomorphism then f^t is not an isomorphism, and therefore $0 \neq \operatorname{Ext}^{s+i}(\operatorname{Coker} f^t,\ )$, which is the kernel of $\operatorname{Ext}^{s+i}(f^t,\ ) = (\operatorname{Ext}^{s+i}(f,\ ))^t$, shows that $\operatorname{Ext}^{s+i}(f,\ )$ is not a monomorphism for any $i \geq 1$. However, for each $i \geq 1$ we have the exact sequence

$$\operatorname{Ext}^{n_L+s+i}(C,\ ) \longrightarrow \operatorname{Ext}^{s+i}(X,\ ) \xrightarrow{\operatorname{Ext}^{s+i}(f,\ )} \operatorname{Ext}^{s+i}(X,\ )\ .$$

From this it follows that $\operatorname{Ext}^{n_L+s+i}(C,\ ) \neq 0$ for all $i \geq 1$, which contradicts that $\operatorname{pdim} C \leq \infty$. Hence, we deduce that f has to be an isomorphism and therefore $\operatorname{Ext}^{s+i}(f,\ )$ is an isomorphism for all $i \geq 1$. From the exact sequence

$$\operatorname{Ext}^{s+1}(X,\ ) \xrightarrow{\operatorname{Ext}^{s+1}(f,\ )} \operatorname{Ext}^{s+1}(X,\ ) \longrightarrow \operatorname{Ext}^{n_L+s+2}(C,\ ) \longrightarrow \operatorname{Ext}^{s+2}(X,\ ) \xrightarrow{\operatorname{Ext}^{s+2}(f,\ )} \operatorname{Ext}^{s+2}(X,\ )$$

one gets that $\operatorname{Ext}^{n_L+s+2}(C,\ ) = 0$ which shows that $\operatorname{pdim}(C) \leq n_L + s + 1$, where $s = \operatorname{pdim}(P \amalg Q) \leq \operatorname{fin.dim} \Omega^{n_L}(A \amalg B)$. This finishes the proof of this proposition.

Wang's result is now a consequence of the following corollary:

Corollary 4. *Suppose* $0 \to A \to B \to C \to 0$ *is an exact sequence with* $\operatorname{pdim} B < \infty$*, then* $\operatorname{pdim}(B) \leq n_L(A \amalg \Omega C) + \operatorname{fin.dim}(\Omega^{n_L(A \amalg \Omega C)}(A \amalg \Omega C)) + 1$.

Proof: Letting $P_0 \to C$ be a projective cover, one obtains the exact sequence $0 \to \Omega C \to A \amalg P_0 \to B \to 0$, which by the proposition gives the result.

We are now in a position to prove the promised result:

Theorem 5. *Let Λ be an artin algebra and I an ideal such that Λ/I is of finite representation type. Then* $\sup\{\operatorname{pdim} X | \operatorname{pdim} X < \infty,\ I^2X = 0\} < \infty$.

Proof: Let M be an additive generator for the category of Λ-modules annihilated by I, which exists since Λ/I is of finite representation type. Let X be a Λ-module of finite projective dimension with $I^2X = 0$. Consider the exact sequence $0 \to IX \to X \to X/IX \to 0$. The subgroup $\langle M \amalg \Omega M\rangle$ of K is finitely generated and $\langle IX \amalg \Omega(X/IX)\rangle$ is a subgroup of $\langle M \amalg \Omega M\rangle$, hence $n_L(M \amalg \Omega M) \geq n_L(IX \amalg \Omega(X/IX))$. Therefore $\operatorname{pdim} X \leq n_L(IX \amalg \Omega(X/IX)) + \operatorname{fin.dim} \Omega^{n_L(IX \amalg \Omega(X/IX))}(IX \amalg \Omega(X/IX)) + 1 \leq n_L(M \amalg \Omega(M)) + \operatorname{fin.dim}(M \amalg \Omega M \amalg \cdots \amalg \Omega^{n_L(M\amalg \Omega M)+1}(M)) + 1$ which is a universal bound independent of X.

The result of Green& Huisgen-Zimmermann now follows by choosing the ideal I to be the radical, and the result of Dräxler and Happel follows by choosing the ideal I an appropriate power of the radical. Here one is using that if $\operatorname{fin.dim} \Lambda^{\mathrm{op}} < \infty$ then for each nonzero (simple) Λ-module S one have that $\operatorname{Ext}^i_\Lambda(S, \Lambda) \neq 0$ for some $0 \leq i \leq \operatorname{fin.dim} \Lambda^{\mathrm{op}} + 1$. For assume that $\operatorname{fin.dim} \Lambda^{\mathrm{op}} = n < \infty$ and take a minimal projective resolution

$$P_{n+2} \to P_{n+1} \to \cdots \to P_1 \to P_0 \to S \to 0$$

of S and apply $\operatorname{Hom}_\Lambda(\ , \Lambda)$. One then gets the complex

$$0 \to (P_0, \Lambda) \to (P_1, \Lambda) \to \cdots \to (P_{n+1}, \Lambda) \to (P_{n+2}, \Lambda)$$

of projective Λ^{op}-modules where the homology is $\operatorname{Ext}^i(S, \Lambda)$ for $0 \leq i \leq n + 1$. Since the finitistic dimension of $\Lambda^{\mathrm{op}} \leq n$, this complex can not be exact since the first morphism is not a split monomorphism, and therefore $\operatorname{Ext}^i(S, \Lambda) \neq 0$ for some $0 \leq i \leq n + 1$.

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