MA3204 - PROBLEM SHEET 5

We will discuss some of the following problems.

Problem 1. Compute $\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$ using (a) a projective resolution of $\mathbb{Z}/2\mathbb{Z}$ and (b) an injective resolution of $\mathbb{Z}$.

Problem 2. Show that $\mathrm{Ext}^n_R(C, A) = (0)$ for all $n \geq 2$ and for all A and C in $\mathrm{Mod}\, R$ if and only if R is left hereditary.

Problem 3. Let $0 \to K \to P \to C \to 0$ be exact with P a projective module. Show that

$$\mathrm{Ext}^n_R(K, A) \simeq \mathrm{Ext}^{n+1}_R(C, A)$$

for all $n \geq 1$. This is called *dimension shift*.

Problem 4. If A is an abelian group with $mA = A$ for some integer m, then every exact sequence $0 \to A \to E \to \mathbb{Z}/m\mathbb{Z} \to 0$ splits.

Problem 5. If A is a torsion abelian group, then

$$\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{R}/\mathbb{Z}).$$

Problem 6.

(a) Show that E in $\mathrm{Mod}\, R$ is injective if and only if $\mathrm{Ext}^1_R(R/I, E) = (0)$ for all left ideals I in R.

(b) Show that if all cyclic R-modules are projective, then all R-modules are projective.

Problem 7. Let Λ be a finite dimensional algebra over a field with Jacobson radical $\mathfrak{r}$. It is known then that $\mathfrak{r}$ is a nilpotent ideal and that $\Lambda/\mathfrak{r}$ as a left Λ-module is semisimple and all simple Λ-modules are isomorphic to a direct summand of $\Lambda/\mathfrak{r}$. Furthermore a module M is semisimple if and only if $\mathfrak{r}M = (0)$.

(a) Show that any Λ-module M has a filtration in semisimple modules, that is, there is a chain of submodules of M

$$(0) = M_n \subseteq M_{n-1} \subseteq \cdots \subseteq M_2 \subseteq M_1 \subseteq M_0 = M$$

such that M_i/M_{i+1} is semisimple for all $i = 0, 1, \ldots, n-1$. (Consider $\mathfrak{r}^{i+1}M \subseteq \mathfrak{r}^i M$).

(b) Show that left projective global dimension of Λ is given by $\sup\{\mathrm{pd}_\Lambda S \mid S \text{ simple } \Lambda\text{-module}\}$.

Problem 8. Let Λ be a finite dimensional algebra over a field with Jacobson radical $\mathfrak{r}$. It is known that $\mathfrak{r}$ is a nilpotent ideal and that $\Lambda/\mathfrak{r}$ as a left Λ-module is semisimple and all simple Λ-modules are isomorphic to a direct summand of $\Lambda/\mathfrak{r}$. A module M is semisimple if and only if $\mathfrak{r}M = (0)$. Furthermore, any simple Λ-module S is given as $\Lambda e/\mathfrak{r}e$ for some idempotent e in Λ.

(a) For an idempotent e in Λ show that $\operatorname{Hom}_\Lambda(e\Lambda, \Lambda) \simeq \Lambda e$ as left Λ-modules.

(b) Let R be a ring with an ideal $I \subseteq R$, and let X and M be a right and a left R-module. Assume that $IM = (0)$. Show that $X \otimes_R M \simeq X/XI \otimes_R M$.

(c) Let P be a finitely generated projective left Λ-module. Show that we have a natural isomorhism

$$\operatorname{Hom}_\Lambda(P, M) \simeq \operatorname{Hom}_\Lambda(P, \Lambda) \otimes_\Lambda M.$$

(d) Consider the additive contravariant functor

$$D = \operatorname{Hom}_k(-, k)\colon \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda^{\mathrm{op}}.$$

This functor sends the simple right Λ-module $S_e^{\mathrm{op}} = e\Lambda/e\mathfrak{r}$ to the simple left Λ-module $S_e = \Lambda e/\mathfrak{r}e$, that is, $D(S_e^{\mathrm{op}}) \simeq S_e$. Recall that $\operatorname{Hom}_\Lambda(X, D(Y)) \simeq D(Y \otimes_\Lambda X)$ by the adjunction isomorphism. Use the above to show that

$$\operatorname{Ext}^1_\Lambda(S_e, S_f) \simeq D(f\mathfrak{r}e/f\mathfrak{r}^2 e)$$

for idempotents e and f in Λ.

(e) Let R be a ring with an ideal $I \subseteq R$. Show that if P is a projective left R-module, then $R/I \otimes_R P$ is a projective left R/I-module.

(f) Let $\Lambda = kQ/I$, where $J^t \subseteq I \subseteq J^2$ for some positive integer t with $J = \langle \text{arrows} \rangle \subseteq kQ$ for some quiver Q. Recall that in this case Λ is a finite dimensional algebra with Jacobson radical $\mathfrak{r} = J/I$. Fact: All left ideals in kQ are projective kQ-modules (kQ is left hereditary). Show that the sequences

$$0 \to J/I \to kQ/I \to (kQ/I)/(J/I) \to 0$$

and

$$0 \to I/IJ \to J/IJ \to J/I \to 0,$$

are exact sequences, where the all maps are natural projections or inclusions. Moreover, that

$$0 \to I/IJ \to J/IJ \to kQ/I \to (kQ/I)/(J/I) \to 0$$

is the start of a projective resolution of $(kQ/I)/(J/I) \simeq kQ/J$. Using dimension shift and similar arguments as in (d) to show that

$$\operatorname{Ext}^2_\Lambda(\Lambda/\mathfrak{r}, D(\Lambda/\mathfrak{r})) = D(I/(JI + IJ)).$$

(g) Let $\Lambda = kQ/I$ as above. Use the above formulas to find the dimension of

$$\operatorname{Ext}^1_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$$

and the dimension of

$$\operatorname{Ext}^2_\Lambda(\Lambda/\mathfrak{r}, D(\Lambda/\mathfrak{r})).$$

Problem 9.

(a) Let $\Lambda = k[x]/(x^2)$ for a field k, and let $S = k[x]/(x)$. Compute $\operatorname{Ext}^i_\Lambda(S, S)$ for all $i \geq 0$.

(b) Let $Q\colon\ 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$, and let $\Lambda = kQ$ for a field. Let $T = \Lambda e_1 \amalg \Lambda e_1/\mathfrak{r}^2 e_1 \amalg \Lambda e_2/\mathfrak{r} e_2$. Show that $\operatorname{Ext}^i_\Lambda(T, T) = (0)$ for all $i \geq 1$.

(c) Let $Q\colon\ \alpha \circlearrowright 1 \underset{\gamma}{\overset{\beta}{\rightleftarrows}} 2$, and let $\Lambda = kQ/\langle \gamma\beta - \alpha^2, \beta\alpha\gamma, \beta\gamma \rangle$. Compute $\operatorname{Ext}^i_\Lambda(S_1, S_1)$ for as many i as possible.