MA3204 - PROBLEM SHEET 4

We will discuss some of the following problems.

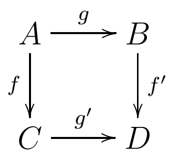
Problem 1. A chain complex $\mathbb{A}\colon \cdots \to A_{n+1} \to A_n \to A_{n-1} \to \cdots$ is said to be *acyclic* if it is exact, i.e. $H_n(\mathbb{A}) = (0)$ for all n.

Show that if $0 \to \mathbb{A}' \to \mathbb{A} \to \mathbb{A}'' \to 0$ is an exact sequence of chain complexes and two of the chain complexes $\mathbb{A}'$, $\mathbb{A}$ and $\mathbb{A}''$ are acyclic, then the third one is also acyclic.

Problem 2. (a) Suppose $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is an exact sequence in $\operatorname{Mod} R$. Let $h\colon B \to X$ be a homomorphism in $\operatorname{Mod} R$, and assume that $hf = 0$. Show that there is a unique homomorphism $t\colon C \to X$ such that $tg = h$.

Dually one can prove, if $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ is exact in $\operatorname{Mod} R$ and $h\colon X \to B$ is such that $gh = 0$, then there is a unique homomorphism $s\colon X \to A$ such that $fs = h$.

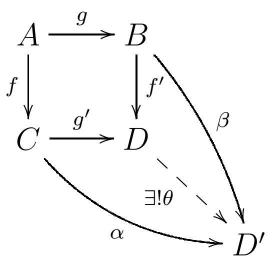
(b) Let $f\colon A \to C$ and $g\colon A \to B$ be two R-homomorphisms. Consider the following diagram



where $D = (C \amalg B)/\{(f(a), -g(a)) \mid a \in A\}$, and $f'(b) = \overline{(0, b)}$ and $g'(c) = \overline{(c, 0)}$.

(i) Show that the diagram is commutative.

(ii) Use (a) to show that if there are homomorphisms $\alpha\colon C \to D'$ and $\beta\colon B \to D'$ such that $\alpha f = \beta g$, then there is a unique homomorphism $\theta\colon D \to D'$ such that $\theta f' = \beta$ and $\theta g' = \alpha$. That is, we have the following commutative diagram.



(*Hint*: $A \xrightarrow{\binom{f}{-g}} C \amalg B \xrightarrow{(g'\ f')} D \to 0$ is exact.)

(iii) Show that $\operatorname{Coker} g \simeq \operatorname{Coker} g'$ (and similarly $\operatorname{Coker} f \simeq \operatorname{Coker} f'$). Show that if g is one-to-one, then g' is one-to-one.

The above diagram defines the *pushout* of the homomorphisms f and g.

(c) Consider the diagram

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{n\cdot -} & \mathbb{Z} \\ {\scriptstyle p}\downarrow & & \\ \mathbb{Z}/(m) & & \end{array}$$

for some integers $m, n \geq 2$, where p is the natural projection and $n \cdot -$ is map given by multiplication by n. Show that the pushout of this diagram gives $D \simeq \mathbb{Z}/(mn)$.

Problem 3. Let $Q = (Q_0, Q_1, h, t)$ be a quiver, that is,

$$\begin{aligned} Q_0 &= \text{the set of vertices} = \{1, 2, \ldots, n\} \\ Q_1 &= \text{the set of arrows (a finite set)} \end{aligned}$$

The functions $h, t\colon Q_1 \to Q_0$ are defined for an arrow $\alpha\colon i \to j$ by setting $t(\alpha) = i$ and $h(\alpha) = j$. Paths in the quiver Q comes in two types, (i) trivial paths $\{e_i\}_{i \in Q_0}$ and (ii) compositions of arrows, $\alpha_n \alpha_{n-1} \cdots \alpha_2 \alpha_1$ where $h(\alpha_i) = t(\alpha_{i+1})$. The set of paths union 0 is endowed with a multiplication given by concatenation, where $e_i \alpha = \alpha$ and $\alpha e_j = \alpha$ if $h(\alpha) = i$ and $t(\alpha) = j$, respectively, and zero otherwise. Furthermore, $e_i e_j = e_i$ if $i = j$ and zero otherwise.

View Q as a category in the following way. Objects in Q are the set of vertices Q_0 and the morphisms $\operatorname{Hom}_Q(i, j) =$ the paths from i to j in Q, where the $1_i = e_i$. Composition of morphisms is induced by the multiplication of paths.

Let $Q = \; 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$. Let $F\colon Q \to \operatorname{mod} k$ be a functor, where k is a field ($\operatorname{mod} k =$ finite dimensional vectors spaces over k). Describe F in terms of vector spaces and linear maps.

Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two functors. Recall that a morphism of functors from F to G, $\varphi\colon F \to G$, is a family of morphism $\{\varphi_C\}_{C \in \mathcal{C}}$ in $\mathcal{D}$ such that for all morphisms $f\colon C \to C'$ in $\mathcal{C}$ there is a commutative diagram

$$\begin{array}{ccc} F(C) & \xrightarrow{\varphi_C} & G(C) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(g)} \\ F(C') & \xrightarrow{\varphi_{C'}} & G(C') \end{array}$$

Let $F, G\colon Q \to \operatorname{mod} k$ be two functors. Describe all the morphisms from the functor F to the functor G in terms of linear maps.

Problem 4. Show that, for natural numbers n and m, we have:

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = (0) \text{ if and only if } \gcd(m,n) = 1.$$

Problem 5. Let $\Lambda = \left(\begin{smallmatrix} k & k \\ 0 & k \end{smallmatrix}\right)$ where k is a field. Let $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and let $e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Let S be the simple right Λ-module given by $e_1\Lambda / \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\Lambda$. Let T be the simple left Λ-module $\Lambda e_2 / \Lambda \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Note that $e_1 \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) e_2 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$.

Compute $\mathrm{Tor}_n^{\Lambda}(S, T)$.