$\text{MA3204 - PROBLEM SHEET 3}$

We will discuss some of the following problems.

Problem 1. Let R and S be two rings. Let B be a R-S-bimodule, and let C be a left S-module. Recall that the tensor product is associative, that is, $(A \otimes_R B) \otimes_S C$ is naturally isomorphic to $A \otimes_R (B \otimes_S C)$.

(a) Show that if B is a flat R-module and C is a flat S-module, then $B \otimes_S C$ is a flat R-module.
(b) If B is a projective R-module and C is a projective S-module, show that $B \otimes_S C$ is a projective R-module.

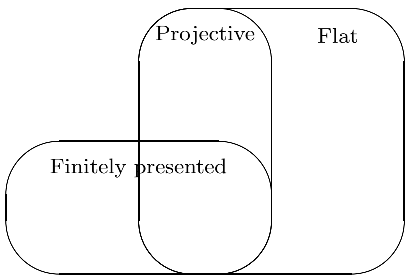
Problem 2. Show that B is flat if $0 \to B' \to B \to B'' \to 0$ is exact and B' and B'' both are flat.

Problem 3. (a) Assume that B in $\operatorname{Mod} R$ is flat and that r in R is a non-zerodivisor. Show that if $rx = 0$ for some x in B, then $x = 0$.

(b) Show that an abelian group is flat if and only if it is torsion free. (Recall that a module M over a domain R is *torsion-free* if $rm = 0$ implies $r = 0$ or $m = 0$, whenever r is in R and m is in M).

Problem 4. Prove that a module Q is a cogenerator for $\operatorname{Mod} R$ if and only if for every R-module M one has that for all elements m in M there exists a R-homomorphism $f\colon M \to Q$ such that $f(m) \neq 0$.

Problem 5. The relationship between finitely presented modules, projective modules and flat modules is give in the following figure:



Give an example of an abelian group in each region of the figure.

Problem 6. Explain that there is an injective homomorphism of groups

$$\phi\colon \mathbb{Z} \to \prod_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z}$$

(p a prime number). Derive from this that

$$\mathbb{Q} \otimes_{\mathbb{Z}} (\prod_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z}) \neq (0).$$

Consequently:

$$\mathbb{Q} \otimes_{\mathbb{Z}} (\prod_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z}) \not\simeq \prod_{n=1}^{\infty} \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z})$$