

# MA3204 - PROBLEM SHEET 2

We will discuss some of the following problems.

Recall that for an  $R$ -homomorphism  $t: X \rightarrow Y$ , we denote  $Y/\text{Im } t$  by  $\text{Coker } t$ . Moreover,  $\text{Ker } t = \{x \in X \mid t(x) = 0\}$ .

Prove Problems 1–4 by diagram chasing.

**Problem 1.** Give a proof of Proposition 2.71.

**Problem 2.** Prove Five Lemma (Proposition 2.72, page 90).

**Problem 3.** Prove the Snake Lemma: Consider the commutative diagram

$$\begin{array}{ccccccc}
 A' & \longrightarrow & A & \xrightarrow{p} & A'' & \longrightarrow & 0 \\
 \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\
 0 & \longrightarrow & C' & \xrightarrow{\iota} & C & \longrightarrow & C''
 \end{array}$$

of modules with exact rows. Show that there is an exact sequence

$$\text{Ker } \alpha \rightarrow \text{Ker } \beta \rightarrow \text{Ker } \gamma \xrightarrow{\partial} \text{Coker } \alpha \rightarrow \text{Coker } \beta \rightarrow \text{Coker } \gamma,$$

where  $\partial: a'' \mapsto \iota^{-1}\beta p^{-1}(a'') + \text{Im } \alpha$ .

**Problem 4.** Prove the  $3 \times 3$ -lemma (Problem 2.32, page 96).

**Problem 5.** Let  $\eta: 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$  be an exact sequence in  $\text{Mod } R$ .

(a) Show that there exists a homomorphism  $f': B \rightarrow A$  such that  $f'f = 1_A$  if and only if there exists a homomorphism  $g': C \rightarrow B$  such that  $gg' = 1_C$ .

If such homomorphisms exist, we say that the sequence  $\eta$  *splits* or is *split exact*. Show that if  $\eta$  is split exact, then  $B \simeq A \amalg C$ .

(b) Assume that  $A$ ,  $B$  and  $C$  are finitely generated  $R$ -modules,  $A$ ,  $B$  and  $C$ , and that  $R$  is a finite dimensional algebra over a field  $k$ . Assume in addition that  $B \simeq A \amalg C$ .

(i) Show that  $\text{Hom}_R(X, Y)$  is a finite dimensional vector space over  $k$  for all finitely generated  $R$ -modules  $X$  and  $Y$ . *Hint:*  $\text{Hom}_R(X, Y) \subseteq \text{Hom}_k(X, Y)$ .

(ii) Given the exact sequence  $\eta: 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ , show that the sequence

$$0 \rightarrow \text{Hom}_R(C, Y) \rightarrow \text{Hom}_R(B, Y) \rightarrow \text{Hom}_R(A, Y) \rightarrow 0$$

is exact for all finitely generated  $R$ -modules  $Y$ .

*Hint:*  $\dim_k \text{Hom}_R(B, Y) = \dim_k \text{Hom}_R(A, Y) + \dim_k \text{Hom}_R(C, Y)$   
for all finitely generated  $R$ -modules  $Y$ .

(iii) Show that the exact sequence  $\eta$  is split exact.

**Challenge.** Let  $f: A \rightarrow B$  and  $g: B \rightarrow C$  be two  $R$ -homomorphisms. Show that there is an exact sequence

$$0 \rightarrow \text{Ker } f \rightarrow \text{Ker } gf \rightarrow \text{Ker } g \rightarrow \text{Coker } f \rightarrow \text{Coker } gf \rightarrow \text{Coker } g \rightarrow 0.$$