MA3204 - PROBLEM SHEET 1

We will discuss some of the following problems.

Problem 1. Show that $G \otimes_{\mathbb{Z}} \mathbb{Q} = (0)$ when G is a finite abelian group.

Problem 2. Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$. Is $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$?

Problem 3. a) Let I be a right ideal and J a left ideal in a ring R. Show that

$$R/I \otimes_R R/J \simeq R/(I + J)$$

as abelian groups, by first finding an isomorphism

$$R/I \otimes_R M \simeq M/IM$$

for a left R-module M.

b) What is $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n$ for arbitrary natural numbers m and n?

Problem 4. Let V and W be vector spaces over a field k with bases $\mathcal{B} = \{v_1, \dots, v_m\}$ and $\mathcal{C} = \{w_1, \dots, w_n\}$.

a) Explain why $V \otimes_k W$ is a vector space with basis $\{v_i \otimes w_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$. What is $\dim_k V \otimes_k W$?

b) Let $A = (a_{ij})$ be a $m \times n$-matrix with a_{ij} in k. Explain why $f \colon V \times W \to k$ is bilinear when f is given by $f(v, w) = x^T A y$, where x and y are the coordinate vectors of v and w with respect to $\mathcal{B}$ and $\mathcal{C}$. Describe the corresponding linear map $F \colon V \otimes_k W \to k$.

Problem 5. Let A be in $\operatorname{Mod} R^{\text{op}}$ and B in $\operatorname{Mod} R$. Let I be a two-sided ideal in R such that $I \subset \operatorname{Ann}(A) \cap \operatorname{Ann}(B)$ (such that A and B are R/I-modules).

Explain why $A \otimes_R B \simeq A \otimes_{R/I} B$.

Problem 6. Recall that a morphism $f \colon A \to B$ is a *split monomorphism* (a *split epimorphism*, and an *isomorphism*) if there ia a morphism $g \colon B \to A$ such that $gf = 1_A$ ($fg = 1_B$; and $gf = 1_A$ and $fg = 1_B$, respectively). Show that any functor preserves split monomorphisms, split epimorphisms and isomorphisms. What is the corresponding statement for contravariant functors?

Challenge. Suppose $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{C}$ is an adjoint pair (F, G) of functors. Let $\eta_C \colon C \to GF(C)$ be the unit of the adjunction, that is, $\eta_C = \tau_{C,F(C)}(1_{F(C)})$ where $\tau_{C,D} \colon \operatorname{Hom}_{\mathcal{D}}(F(C), D) \to \operatorname{Hom}_{\mathcal{C}}(C, G(D))$ is the adjunction bijections. Show that for any morphism $f \colon C \to G(D)$ in $\mathcal{C}$, there is a morphism $h \colon GF(C) \to G(D)$ in $\mathcal{C}$ such that $f = h\eta_C$.